Integrable discrete equations: some recent results

V.E. Adler

27th April 2005

Recent classification results are presented:

- for the discrete KdV type equations (quad-equations) [1]
- for the Yang-Baxter mappings [2]
- for the discrete Toda lattice type equations [3, 4]

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[1] V.E. Adler, A.I. Bobenko, Yu.B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. *Comm. Math. Phys.* **233** (2003) 513–543.

[2] V.E. Adler, A.I. Bobenko, Yu.B. Suris. Geometry of Yang-Baxter maps: pencils of conics and quadrirational mappings. *Comm. Anal. and Geom.* **12:5** (2004) 967–1007.

[3] V.E. Adler. On the structure of the Bäcklund transformations for the relativistic lattices. *J. Nonl. Math. Phys.* **7:1** (2000) 34–56.

[4] V.E. Adler, Yu.B. Suris. Q4: Integrable master equation related to an elliptic curve. *Int. Math. Res. Not.* **47** (2004) 2523–2553.

[5] A.I. Bobenko, Yu.B. Suris. Discrete differential geometry. Consistency as integrability. *math.DG/0504358*.

Contents

1 Introduction

1.1 Types of equations

In the simplest situation we consider equations defined on the lattice $\mathbb{Z}^2$. Quad-equations are equations of the form

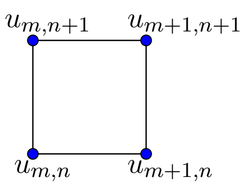
$$Q_{m,n}(u_{m,n}, u_{m+1,n}, u_{m,n+1}, u_{m+1,n+1}) = 0. \tag{1}$$

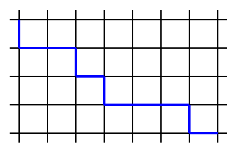
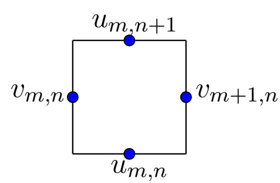
The variables u are associated to the vertices of the square lattice. The equation must be solvable with respect to any of 4 unknowns.

The general form of Yang-Baxter maps is

$$u_{m,n+1} = f_{m,n}(u_{m,n}, v_{m,n}), \quad v_{m+1,n} = g_{m,n}(u_{m,n}, v_{m,n}). \tag{2}$$

The variables u, v are associated to the edges of the square lattice. The simplest choice of the initial data for both types of equations is along the coordinate axes or on the "staircase".



Discrete Toda type lattices are equations of the form

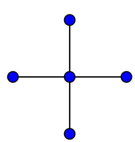
$$f^1_{m,n}(u_{m,n}, u_{m-1,n}) + f^2_{m,n}(u_{m,n}, u_{m+1,n}) + f^3_{m,n}(u_{m,n}, u_{m,n-1}) + f^4_{m,n}(u_{m,n}, u_{m,n+1}) = 0.$$

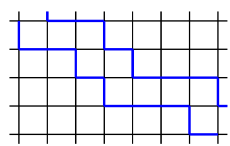
The simplest choice of initial data is on the pair of lines $n = 0$, $n = 1$.

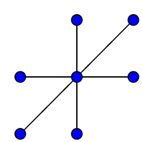
Discrete relativistic Toda type lattices:

$$f^1_{m,n}(u_{m,n}, u_{m-1,n}) + f^2_{m,n}(u_{m,n}, u_{m+1,n}) + f^3_{m,n}(u_{m,n}, u_{m,n-1}) + f^4_{m,n}(u_{m,n}, u_{m,n+1}) \\ + f^5_{m,n}(u_{m,n}, u_{m-1,n-1}) + f^6_{m,n}(u_{m,n}, u_{m+1,n+1}) = 0.$$

The simplest choice of initial data is on the double staircase.







1.2 Planar graphs and quad-graphs

Integrability of the equations (1), (2) (accordingly to the definition below) is a local property. In particular, analogous equations can be considered not only on $\mathbb{Z}^2$, but also on arbitrary quad-graph, that is cellular decomposition of the plane with quadrilateral cells. On the correct choice of initial data for such equations see [8].

Analogously, discrete Toda lattices can be defined as equations on "stars" for arbitrary planar graph G:

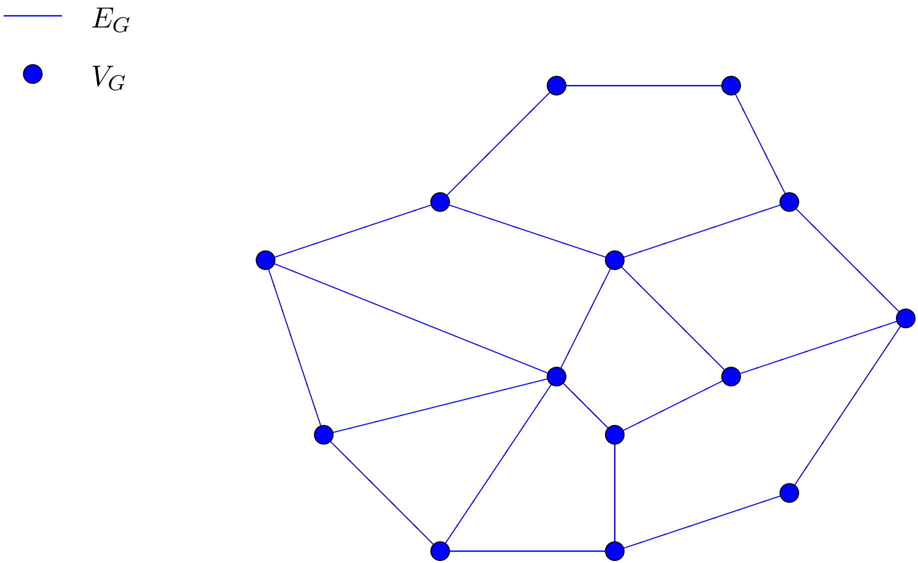
$$\sum_{j:(i,j)\in E_G} f_{ij}(u_i, u_j) = 0$$

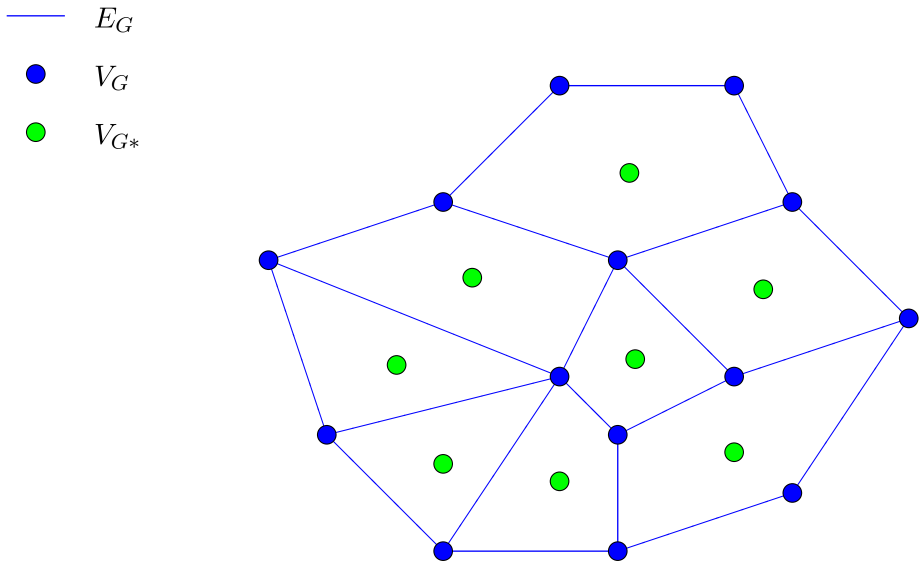
Remind, that the graph G can be associated to the bipartite quad-graph Q, such that

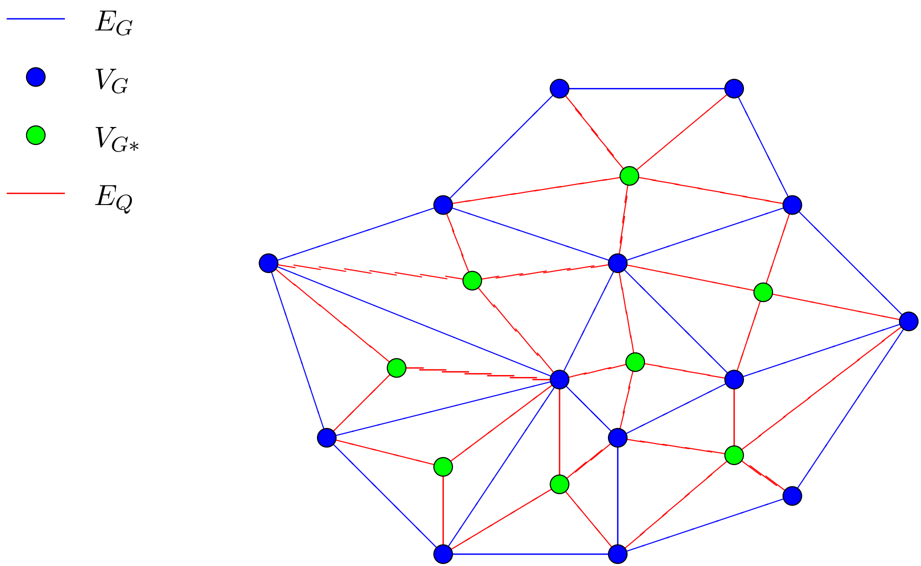
$$V_Q = V_G \cup V_{G^*}, \qquad E_Q = \{(i,i^*)|\ i \in V_G,\ i^* \in V_{G^*},\ i \in f(i^*)\}$$

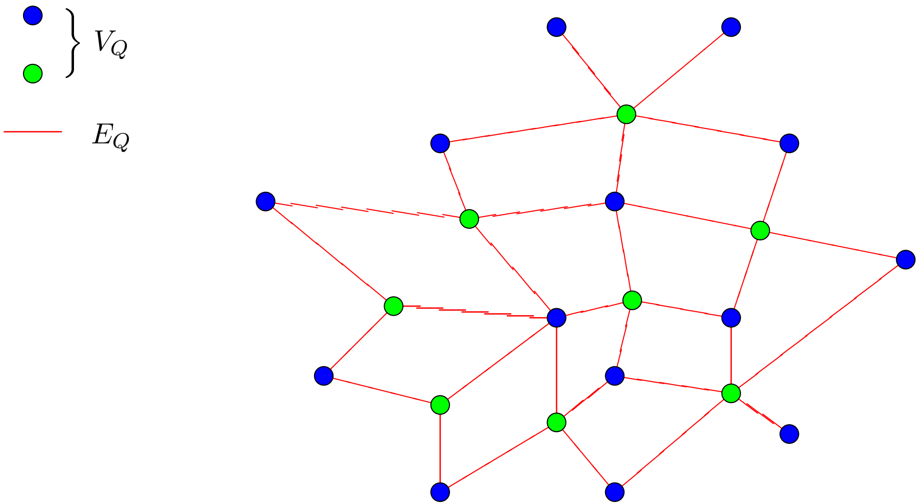
where $f(i^*)$ is the face of G, corresponding to the vertex i^* of dual graph.

[6] V.E. Adler. Discrete equations on planar graphs. *J. Phys. A* **34** (2001) 10453–10460.
[7] A.I. Bobenko, Yu.B. Suris. Integrable systems on quad-graphs. *Int. Math. Res. Notices* **11** (2002) 573–611.
[8] V.E. Adler, A.P. Veselov. Cauchy problem for integrable discrete equations on quad-graphs. *Acta Appl. Math.* **84** (2004) 237–262.


E_G
V_G

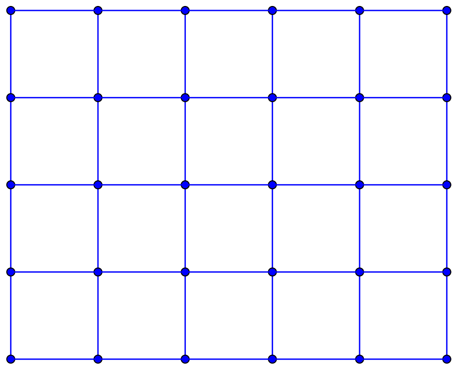

E_G
V_G
V_{G*}


E_G
V_G
V_{G*}
E_Q


V_Q
E_Q

In other words, the edges of the graph G are diagonals of the faces of quad-graph Q joining the vertices of one of two types. In the next Section we will see that this correspondence can be prolonged on quad-equations and discrete Toda lattices.

Example 1. Square lattice.



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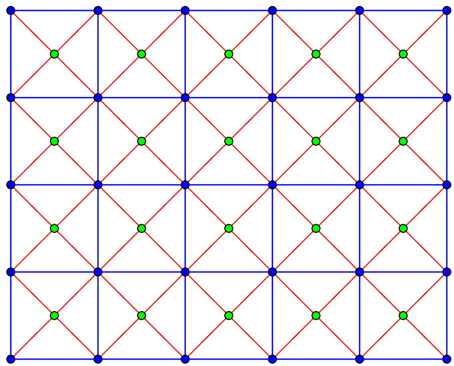

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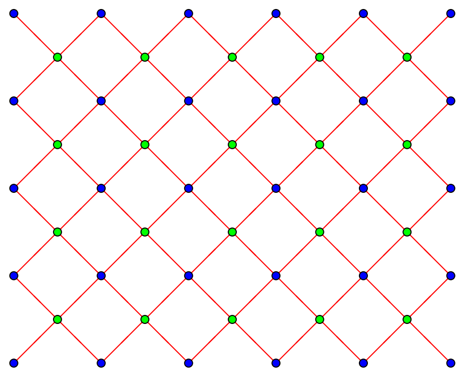
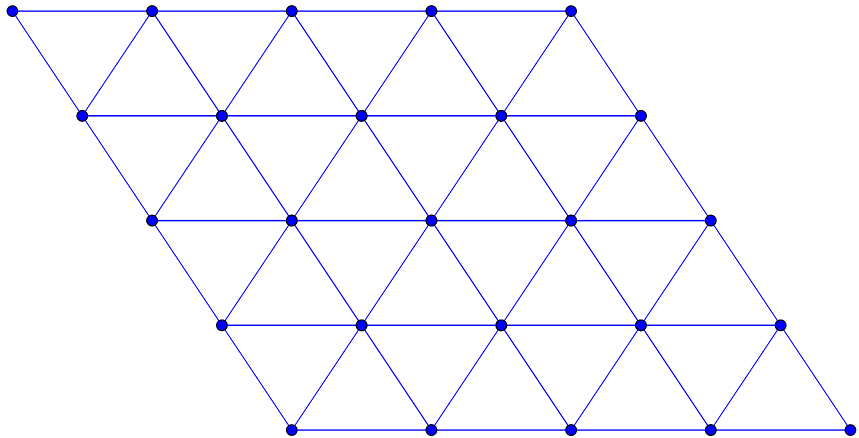
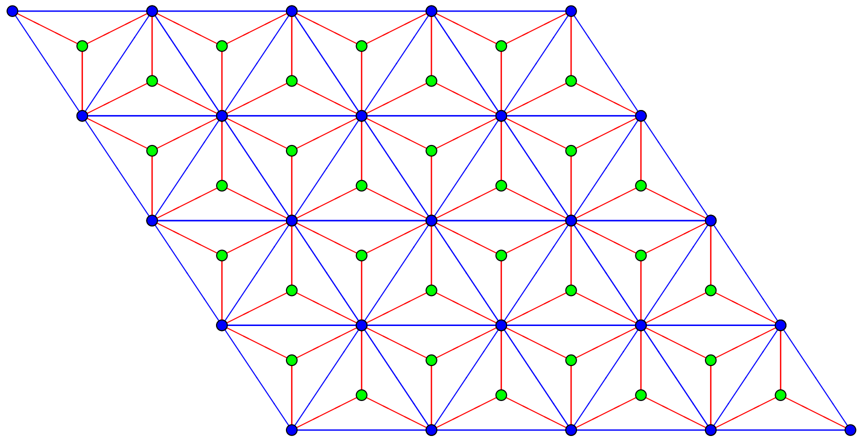


Example 2. Triangle lattice.



Example 2. Triangle lattice.



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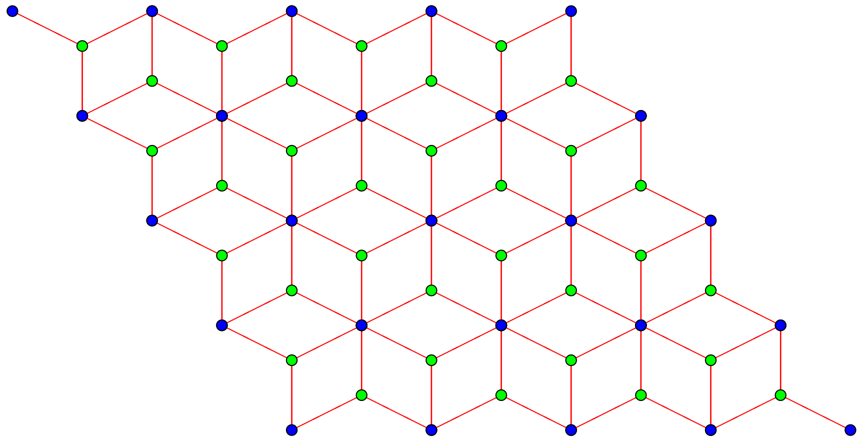


1.3 Relation between the different types of equations: KdV example

- Potential KdV equation: $u_t = u_{xxx} - 6u_x^2$
- Bäcklund transformation:

$$u_{1,x} + u_x = (u_1 - u)^2 + a_1, \qquad u_{2,x} + u_x = (u_2 - u)^2 + a_2$$

- Nonlinear superposition principle (quad-equation):

(3) $$(u - u_{12})(u_1 - u_2) = a_1 - a_2$$

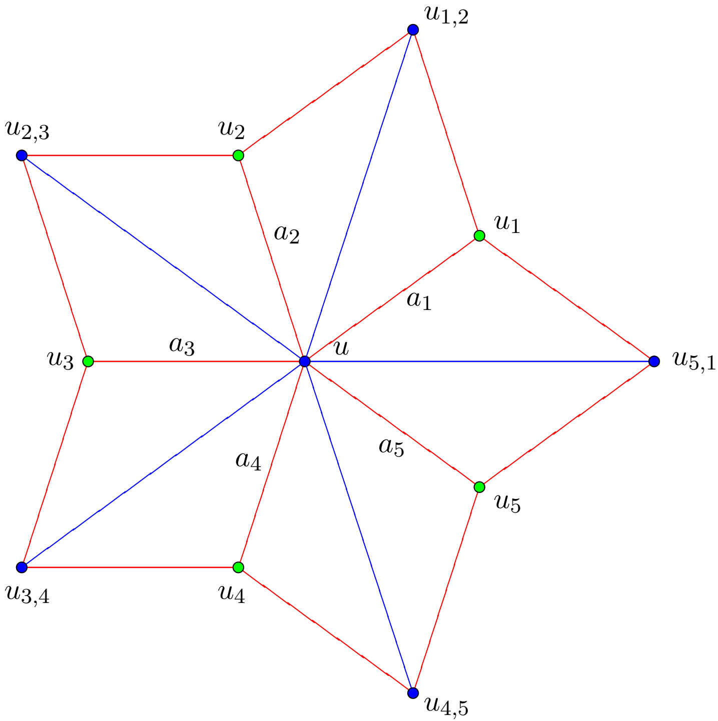
Parameters a are associated to the edges of quad-graph. The parameters corresponding to the opposite edges of a cell coincide.

- Yang-Baxter map ($v = u_1 - u$, $w = u_2 - u$):

$$v_2 = -w + \frac{a_1 - a_2}{w - v}, \qquad w_1 = -v + \frac{a_1 - a_2}{w - v}$$

- Toda lattice (the sum is over the cells incident to the vertex u):

$$\sum_n \frac{a_n - a_{n+1}}{u_{n,n+1} - u} = 0$$


$u_{1,2}$
$u_{2,3}$
u_2
a_2
u_1
a_1
a_3
u_3
u
$u_{5,1}$
a_5
a_4
u_5
$u_{3,4}$
u_4
$u_{4,5}$

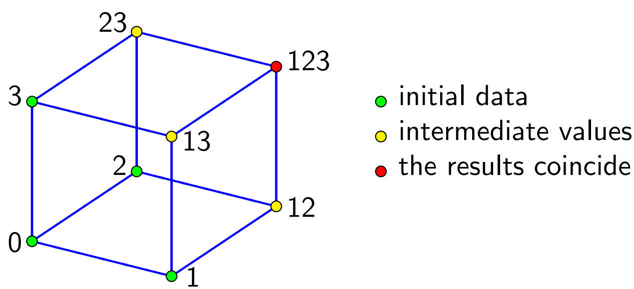
2 Quad-equations

2.1 $3D$-consistency

Denote the vertices of the cube as shown below and consider the system of 6 quad-equations associated to the faces of the cube (assuming $u_{ij} := u_{ji}$):

$$Q_{ij}(u, u_i, u_j, u_{ij}) = 0, \quad Q_{ij}(u_k, u_{ik}, u_{jk}, u_{123}) = 0.$$

This system is called $3D$-consistent [9, 7] if the values u_{123} calculated in three possible ways coincide for any choice of initial data u, u_1, u_2, u_3.



[9] F.W. Nijhoff, A.J. Walker. The discrete and continuous Painlevé hierarchy and the Garnier system. *Glasgow Math. J.* **43A** (2001) 109–123.

Example 3. Discrete KdV equation (3).

$$(u-u_{ij})(u_i-u_j)=a_i-a_j,\quad (u_k-u_{123})(u_{ik}-u_{jk})=a_i-a_j.$$

(Parameter a_i corresponds to 4 edges of the cube parallel to the edge $(0,i)$.) One of the ways of computation yields

$$u_{12}=u-\frac{a_1-a_2}{u_1-u_2},\quad u_{13}=u-\frac{a_1-a_3}{u_1-u_3},$$
$$u_{123}=u_1-\frac{a_2-a_3}{u_{12}-u_{13}}=\frac{a_1u_1(u_2-u_3)+a_2u_2(u_3-u_1)+a_3u_3(u_1-u_2)}{a_1(u_2-u_3)+a_2(u_3-u_1)+a_3(u_1-u_2)}.$$

Since this expression is symmeric with respect to the subscripts, two another ways give the same result.

Example 4. Linear equation.

$$u_{ij}-u_i-u_j+u=0,\quad u_{123}-u_{ik}-u_{jk}+u_k=0.$$

Independently on the order of computations

$$u_{123}=u_1+u_2+u_3-2u.$$

2.2 Classification theorem

The classification of $3D$-consistent equations is obtained in [1] under the following assumptions:

- $Q_{ij}(u,u_i,u_j,u_{ij}) = Q(u,u_i,u_j,u_{ij},a_i,a_j)$ where a_i are parameters assigned to the edges parallel to $(0,i)$.
- Function Q is affinne-linear polynomial in u, with coefficients depending on a:

$$Q = c_1uu_1u_2u_{12} + \cdots + c_{16}.$$

- Equations admit the symmetry group of the square ($\varepsilon^2 = \sigma^2 = 1$):

$$(4) \qquad Q(u,u_1,u_2,u_{12},a_1\alpha_2) = \varepsilon Q(u,u_2,u_1,u_{12},a_2,a_1) = \sigma Q(u_1,u,u_{12},u_2,a_1,a_2)$$

- The tetrahedron condition is satisfied: u_{123} as the function on initial data does not depend on u. (Cf. examples 3,4).

The proof is based on the following important correspondence between affine-linear equations on 4 variables, biquadratic polynomials on 2 variables and 4-th degree polynomials on 1 variable:

$$Q(u,v,w,x) \quad \to \quad g(u,v)=Q_wQ_x-QQ_{wx} \quad \to \quad r(u)=g_v^2-2gg_{vv}.$$

This correspondence is invariant with respect to the Möbius transformations.

It can be proved that under the imposed assumptions the relation holds

$$Q_{u_2}Q_{u_{12}}-QQ_{u_2u_{12}}=k(a_1,a_2)h(u,u_1,a_1)$$

where

$$k(a_2,a_1)=-k(a_1,a_2), \quad h(u_1,u,a_1)=h(u,u_1,a_1)$$

and moreover, the biquadratic h is such that the polynomial

$$h_{u_1}^2-2hh_{u_1u_1}=r(u)$$

does not depend on parameters at all. After this, the classification is reduced to the problem of reconstruction of h and Q starting from the polynomial r which can be bringed to some canonical form by Möbius transformations.

Theorem 1. Up to the simultaneous Möbius transformations of variables and point transformations of parameters $3D$-consistent equations satisfying the above assumptions are exhausted by the following list:

$$(\mathrm{Q}_1) \qquad a_1(u-u_2)(u_1-u_{12})-a_2(u-u_1)(u_2-u_{12})=\delta^2a_1a_2(a_2-a_1)$$

$$(\mathrm{Q}_2) \qquad a_1(u-u_2)(u_1-u_{12})-a_2(u-u_1)(u_2-u_{12}) \\ +a_1a_2(a_1-a_2)(u+u_1+u_2+u_{12})=a_1a_2(a_1-a_2)(a_1^2-a_1a_2+a_2^2)$$

$$(\mathrm{Q}_3) \qquad (a_2^2-a_1^2)(uu_{12}+u_1u_2)+a_2(a_1^2-1)(uu_1+u_2u_{12})-a_1(a_2^2-1)(uu_2+u_1u_{12}) \\ =\delta^2(a_1^2-a_2^2)(a_1^2-1)(a_2^2-1)/(4a_1a_2)$$

$$(\mathrm{Q}_4) \qquad \operatorname{sn}a_1\operatorname{sn}a_2\operatorname{sn}(a_1-a_2)(k^2uu_1u_2u_{12}+1)+\operatorname{sn}a_1(uu_1+u_2u_{12}) \\ -\operatorname{sn}a_2(uu_2+u_1u_{12})-\operatorname{sn}(a_1-a_2)(uu_{12}+u_1u_2)=0, \qquad \operatorname{sn}a\equiv\operatorname{sn}(a;k)$$

$$(\mathrm{H}_1) \qquad (u-u_{12})(u_1-u_2)=a_1-a_2$$

$$(\mathrm{H}_2) \qquad (u-u_{12})(u_1-u_2)+(a_2-a_1)(u+u_1+u_2+u_{12})=a_1^2-a_2^2$$

$$(\mathrm{H}_3) \qquad a_1(uu_1+u_2u_{12})-a_2(uu_2+u_1u_{12})=\delta(a_2^2-a_1^2)$$

$$(\mathrm{A}_1) \qquad a_1(u+u_2)(u_1+u_{12})-a_2(u+u_1)(u_2+u_{12})=\delta^2a_1a_2(a_1-a_2)$$

$$(\mathrm{A}_2) \qquad (a_2^2-a_1^2)(uu_1u_2u_{12}+1)=a_1(a_2^2-1)(uu_1+u_2u_{12})-a_2(a_1^2-1)(uu_2+u_1u_{12})$$

- Eq (A_1) is reduced to (Q_1) by the change $u_i \to -u_i$; (A_2) is reduced to (Q_3) by the change $u_i \to 1/u_i$.
- Eqs (Q_1)–(Q_3) and (H_1), (H_2) can be obtained from (Q_4), (H_3) by degenerations and as limiting cases.
- Eq (Q_4) defines the nonlinear superposition of BTs for the Krichever-Novikov eq [10]

$$u_t = u_{xxx} - \frac{3(u_{xx}^2 - r(u))}{2u_x}, \quad r^{(5)} = 0 \tag{5}$$

- The given form of (Q_4) is found by Hietarinta [SIDE-2004 talk]. In [1] this equation was presented in much more cumbersome form related to the Weierstrass form of elliptic curve $A^2 = r(a) = 4a^3 - g_2 a - g_3$.
- The problem of classification without additional assumptions (affine-linearity, prescribed dependence on parameters, symmetry, tetrahedron property) remains open. In particular, several examples without tetrahedron property were found in [11]. It can be proved that the biquadratics h corresponding to such equations are reducible.
- Several equations are known with polynomial Q quadratic in each variable, but all these examples can be reduced to affine-linear ones by Miura type transformations.

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[10] I.M. Krichever, S.P. Novikov. Holomorphic bundles over algebraic curves and nonlinear equations. *Uspekhi Mat. Nauk* **35:6** (1980) 47–68.

[11] J. Hietarinta. A new two-dimensional lattice model that is "consistent around a cube". *J. Phys. A* **37:6** (2004) L67–73.

2.3 Zero curvature representation

An affine-linear equation $Q=0$ may be interpreted as Möbius transformation between any pair of variables, with coefficients depending on the rest pair. Let

$$u_{13}=M(u_1,u,a_1,a_3;u_3)=\frac{Au_3+B}{Cu_3+D}$$

then

$$u_{23}=M(u_2,u,a_2,a_3;u_3),\quad u_{123}=M(u_{12},u_2,a_1,a_3;u_{23})=M(u_{12},u_1,a_2,a_3;u_{13}).$$

Since the composition of Möbius transformations corresponds to the product of the matrices, hence denoting $a_3\to\lambda$ and introducing the normalization factor yields the zero curvature representation

$$L(u_{12},u_1,a_2,\lambda)L(u_1,u,a_1,\lambda)=L(u_{12},u_2,a_1,\lambda)L(u_2,u,a_2,\lambda)$$

with the matrix

$$L(u_1,u,a_1,\lambda)=(AD-BC)^{-1/2}\begin{pmatrix}A & B\\ C & D\end{pmatrix}.$$

For example, in the case of the discrete KdV equation (H1) one obtains

$$L(u_1,u,a_1,\lambda)=\begin{pmatrix}u & -uu_1+a_1-\lambda\\ 1 & -u_1\end{pmatrix}.$$

2.4 Three-leg form and discrete Toda lattices

Definition 1. Let the quad-equation $Q(u, u_1, u_2, u_{12}, a_1, a_2) = 0$ possesses the square symmetry (4). We will say that it admits three-leg form if it is equivalent to the equation of the form

$$\phi(u, u_{12}, a_1, a_2) = \psi(u, u_1, a_1) - \psi(u, u_2, a_2).$$

Any three-leg equation corresponds to a discrete Toda lattice on the planar graph

$$\sum_n \phi(u, u_{n,n+1}, a_n, a_{n+1}) = 0$$

where the sum is taken over the edges incident to the vertex u.

Three-leg form exists for all equations from the above list. The general formula can be proved

$$\psi(u, u_1, a_1) = \int \frac{du_1}{h(u, u_1, a_1)} + C(u, a_1).$$

For the equations (Q_n), a point change of parameters $a = a(\alpha)$ exists such that $\phi(u, u_{12}, a_1, a_2) = \psi(u, u_{12}, a(\alpha_1 - \alpha_2))$. Moreover, it is often covenient to make a change of the variables $u = u(x)$ as well and use the multiplicative three-leg form

$$F(x, x_{12}, \alpha_1 - \alpha_2) = F(x, x_1, \alpha_1)/F(x, x_2, \alpha_2).$$

	$F(x,y,\alpha)$	$u=u(x)$	$a=a(\alpha)$
$(\mathrm{Q}_1)_{\delta=0}$	$\exp(\alpha/(x-y))$	x	α
$(\mathrm{Q}_1)_{\delta=1}$	$\dfrac{x-y+\alpha}{x-y-\alpha}$	x	α
(Q_2)	$\dfrac{(x+y+\alpha)(x-y+\alpha)}{(x+y-\alpha)(x-y-\alpha)}$	x^2	α
$(\mathrm{Q}_3)_{\delta=0}$	$\dfrac{\sinh(x-y+\alpha)}{\sinh(x-y-\alpha)}$	$\exp 2x$	$\exp 2\alpha$
$(\mathrm{Q}_3)_{\delta=1}$	$\dfrac{\sinh(x+y+\alpha)\sinh(x-y+\alpha)}{\sinh(x+y-\alpha)\sinh(x-y-\alpha)}$	$\cosh 2x$	$\exp 2\alpha$
(Q_4)	$\dfrac{\mathrm{sn}(x+\alpha)-\mathrm{sn}\,y}{\mathrm{sn}(x-\alpha)-\mathrm{sn}\,y}\cdot\dfrac{\Theta_4(x+\alpha)}{\Theta_4(x-\alpha)}$	$\mathrm{sn}\,x$	α

$$(\mathrm{H}_1):\quad \frac{a_1-a_2}{u-u_{12}}=u_1-u_2,\qquad (\mathrm{H}_2):\quad \frac{u-u_{12}+a_1-a_2}{u-u_{12}-a_1+a_2}=\frac{u+u_1+a_1}{u+u_2+a_2}$$

$$(\mathrm{H}_3):\quad \frac{a_2u-a_1u_{12}}{a_1u-a_2u_{12}}=\frac{uu_1+\delta a_1}{uu_2+\delta a_2}$$

Remark. For the eq (Q_4) with the polynomial r in Weierstrass form, the leg is

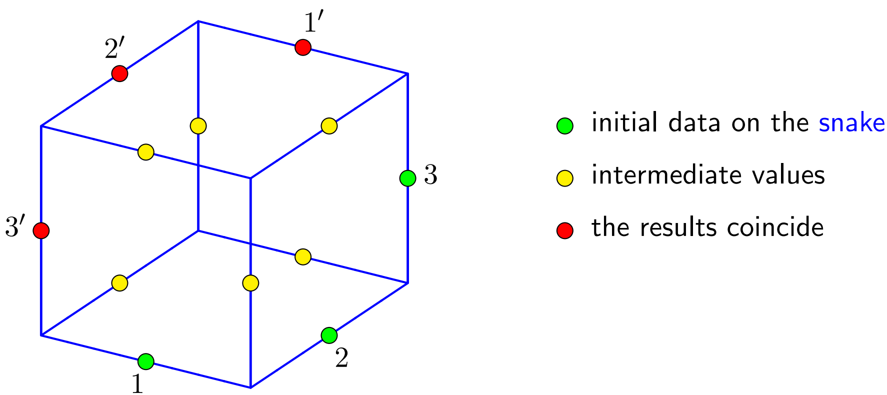
$$F=\frac{\sigma(x+y+\alpha)\sigma(x-y+\alpha)}{\sigma(x+y-\alpha)\sigma(x-y-\alpha)}.$$

3 Yang-Baxter maps

Consider mappings $R_{ij} : C_i \times C_j \to C_i \times C_j$ where C_i are some spaces or manifolds. Let the mapping $\widehat{R}_{ij} : C_1 \times C_2 \times C_3 \to C_1 \times C_2 \times C_3$ acts as R_{ij} on i-th and j-th factors and be identical on the rest one.

Definition 2. R_{ij} are called Yang-Baxter mappings if

$$\widehat{R}_{23} \circ \widehat{R}_{13} \circ \widehat{R}_{12} = \widehat{R}_{12} \circ \widehat{R}_{13} \circ \widehat{R}_{23}$$



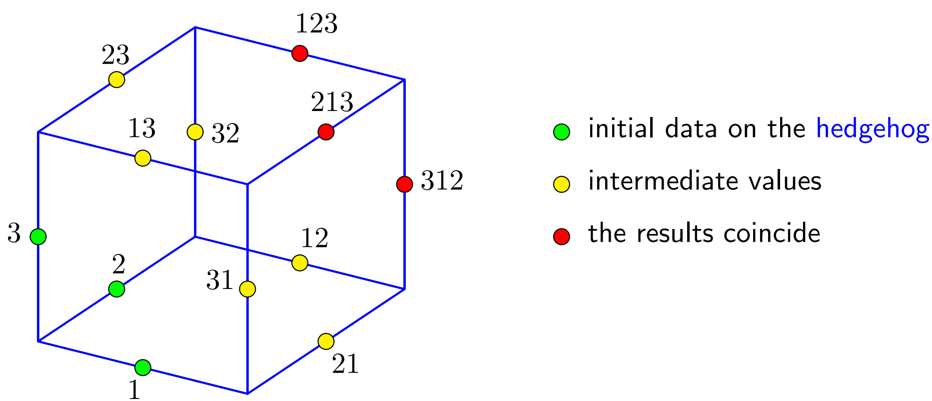
We will use the following equivalent definition.

Let $F_{ij} : C_i \times C_j \to C_i \times C_j$ and

$$F_{ij} : (X_i, X_j) \mapsto (X_{ij}, X_{ji}), \quad F_{ij} : (X_{ik}, X_{jk}) \mapsto (X_{ikj}, X_{jki}).$$

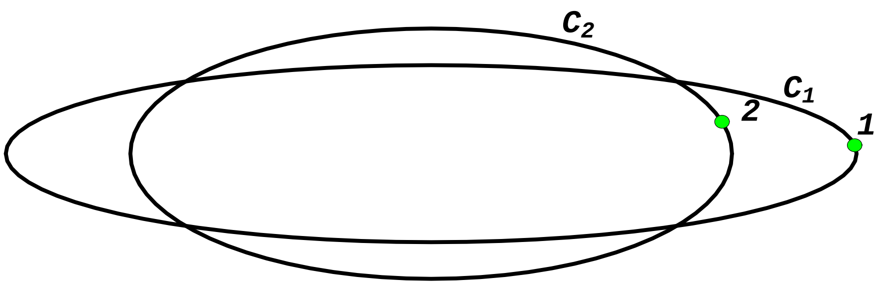
Definition 3. The mappings F_{ij} are called 3D-consistent if

$$X_{ijk} \equiv X_{ikj}$$



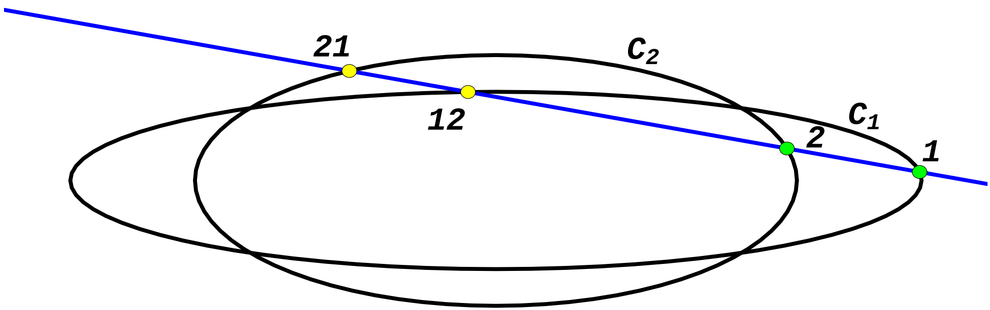
3.1 Yang-Baxter mappings on the linear pencils of conics

Let X_1, X_2 be points on the conic sections C_1, C_2 respectively.

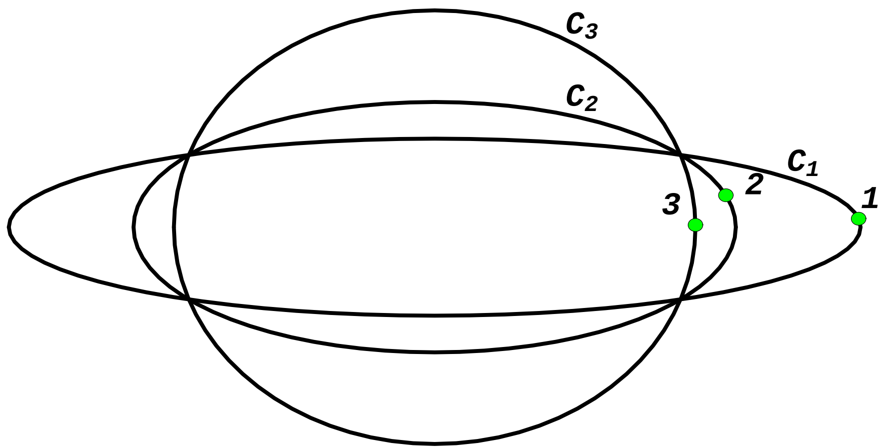


The mapping $F_{12} : C_1 \times C_2 \to C_1 \times C_2$ is defined as follows:

$$X_{12} = X_1 X_2 \cap C_1, \quad X_{21} = X_1 X_2 \cap C_2$$

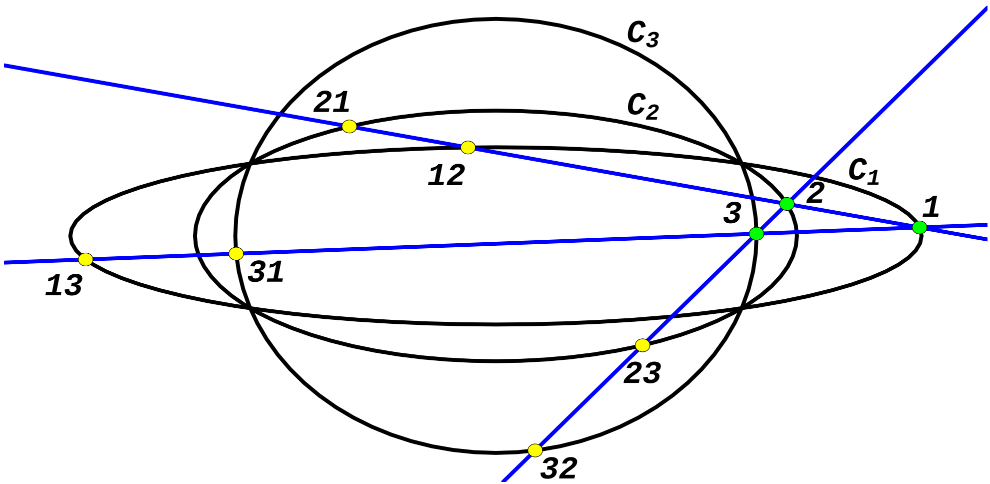


Consider the initial data on three conics from the linear pencil.



Consider the initial data on three conics from the linear pencil.

Apply the mappings $F_{ij} : (X_i, X_j) \mapsto (X_{ij}, X_{ji})$.

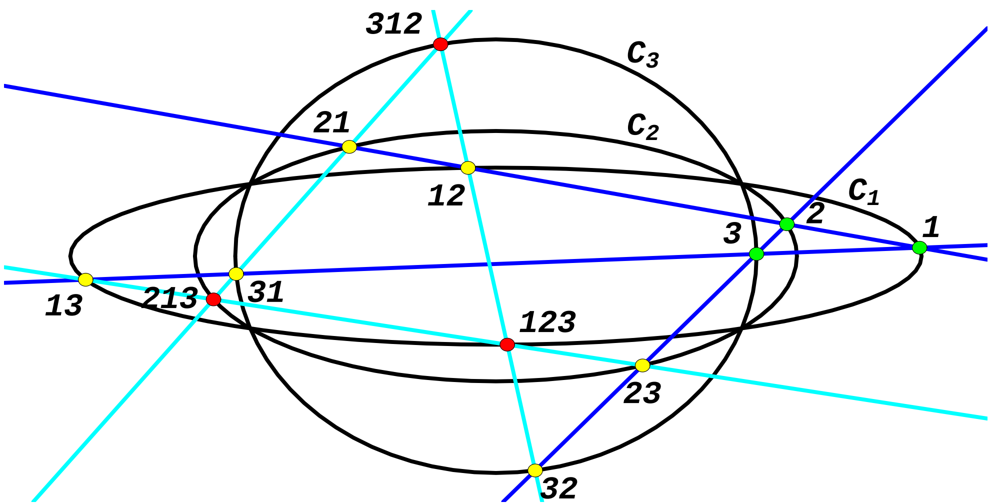


Consider the initial data on three conics from the linear pencil.

Apply the mappings $F_{ij} : (X_i, X_j) \mapsto (X_{ij}, X_{ji})$.

Apply the mappings once more. Let $F_{ij} : (X_{ik}, X_{jk}) \mapsto (X_{ikj}, X_{jki})$.

Theorem 2. The mappings F_{ij} are $3D$-consistent: $X_{ijk} = X_{ikj}$.



Under a rational parametrization of the conics $C_i : X_i = X_i(x_i)$ the mapping F_{12} turns into a birational mapping on $\mathbb{CP}^1 \times \mathbb{CP}^1$. There exist 5 projective types of the linear pencils of conics $C_i = C + a_i K$ [12]. These types lead to the following list of the mappings $(i, j \in \{1, 2\})$:

$$
\begin{aligned}
x_{ij} &= a_i x_j \frac{(1-a_2)x_1 + a_2 - a_1 + (a_1 - 1)x_2}{a_2(1-a_1)x_1 + (a_1 - a_2)x_2 x_1 + a_1(a_2 - 1)x_2} \\
x_{ij} &= \frac{x_j}{a_i} \cdot \frac{a_1 x_1 - a_2 x_2 + a_2 - a_1}{x_1 - x_2} \\
x_{ij} &= \frac{x_j}{a_i} \cdot \frac{a_1 x_1 - a_2 x_2}{x_1 - x_2} \\
x_{ij} &= x_j \left(1 + \frac{a_2 - a_1}{x_1 - x_2}\right) \\
x_{ij} &= x_j + \frac{a_1 - a_2}{x_1 - x_2}
\end{aligned}
\tag{6}
$$

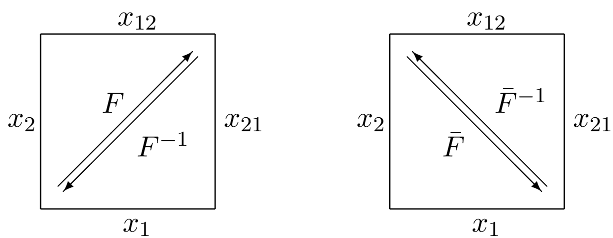
The first one corresponds to the above figures with 4-point locus.

All these mappings can be obtained from those quad-equations listed in Theorem 1, which are invariant with respect to the shift $u \to u + c$ or scaling $u \to cu$, by the changes $x_i = u_i - u$ or $x_i = u_i / u$.

[12] M. Berger. Geometry. *Springer-Verlag, Berlin* 1987.

3.2 Quadrirational mappings

Definition 4 ([13, 2]). The mapping $F : C_1 \times C_2 \to C_1 \times C_2$ is called quadrirational if it and the mappings $F(x_1, \cdot) : C_2 \to C_2$, $F(\cdot, x_2) : C_1 \to C_1$ for almost all $x_i \in C_i$ are birational isomorphisms.



In the case $C_1 = C_2 = \mathbb{CP}^1$, a quadrirational mapping is of the form

$$F : \quad x_{12} = f(x_1, x_2) = \frac{a(x_2)x_1 + b(x_2)}{c(x_2)x_1 + d(x_2)}, \quad x_{21} = g(x_1, x_2) = \frac{A(x_1)x_2 + B(x_1)}{C(x_1)x_2 + D(x_1)}$$

with some special coefficients, such that the mappings F^{-1}, $\bar{F}$,$\bar{F}^{-1}$ be of the same form.

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[13] P. Etingof. Geometric crystals and set-theoretical solutions to the quantum Yang-Baxter equation. *Preprint math.QA/0112278*.

Assuming the nondegeneracy conditions

$$f_{x_1}g_{x_2}-f_{x_2}g_{x_1}\not\equiv 0,\quad f_{x_1}\not\equiv 0,\quad f_{x_2}\not\equiv 0,\quad g_{x_1}\not\equiv 0,\quad g_{x_2}\not\equiv 0,$$

one can prove that the coefficients can be at most quadratic polynomials. Moreover, the mapping F is defined by the pair of polynomial equations

$$P(x_2,x_1,x_{21})=0,\quad Q(x_2,x_{12},x_{21})=0,$$

where either

- P,Q are linear in each argument

or

- P,Q are linear in x_2,x_{21} and quadratic resp. in x_1,x_{12}, and are related by formula

$$Q(x_2,x_{12},x_{21})=(\gamma x_{12}+\delta)^2P\Big(x_2,\frac{\alpha x_{12}+\beta}{\gamma x_{12}+\delta},x_{21}\Big).$$

Theorem 3. Up to the Möbius transformations, all nondegenerate quadrirational mappings, such that $\max\deg(a,b,c,d)=\max\deg(A,B,C,D)=2$, are exhausted by the list (6).

4 Multifield generalizations

4.1 Quad-equations

Classification of multifield quad-equations is hardly possible. One of the reasons is that these equations are not polynomial, in contrast to the scalar case. Probably, the simplest example is the vector analog of the discrete KdV eq:

$$u - u_{12} = \frac{a_1 - a_2}{|u_1 - u_2|^2}(u_1 - u_2).$$

This equation admits an interesting reduction $a_i = -|u_i - u|^2$ [14]. Some other examples can be found in [15, 16].

[14] V.E. Adler. Integrable deformations of a polygon. *Physica D* **87** (1995) 52–57.
[15] A.I. Bobenko, Yu.B. Suris. Integrable non-commutative equations on quad-graphs. The consistency approach. *Lett. Math. Phys.* **61** (2002) 241–254.
[16] W.K. Schief. Isothermic surfaces in spaces of arbitrary dimension: integrability, discretization and Bäcklund transformations. A discrete Calapso equation. *Stud. Appl. Math.* **106** (2001) 85–137.

The nonabelian analogs for the Krichever-Novikov eq (5) exist only for few special cases:

- $r=0$ (Schwarz-KdV). The equation, its BT and NSP are

$$u_{t_3}=u_{xxx}-\frac{3}{2}u_{xx}u_x^{-1}u_{xx},\qquad u_{i,x}=a_i(u-u_i)u_x^{-1}(u-u_i)$$
$$a_1(u-u_2)(u_2-u_{12})^{-1}=a_2(u-u_1)(u_1-u_{12})^{-1}$$

- $r=4$

$$u_{t_3}=u_{xxx}-\frac{3}{2}u_{xx}u_x^{-1}u_{xx}+6u_x^{-1}+3[u_x^{-1},u_{xx}],\qquad u_{i,x}=\frac{1}{a_i}(u-u_i+a_i)u_x^{-1}(u-u_i-a_i)$$
$$a_1(u_1-u_{12}+a_2)(u-u_1-a_1)^{-1}=a_2(u_2-u_{12}+a_1)(u-u_2-a_2)^{-1}$$

- $r=u^2$

$$u_{t_3}=u_{xxx}-\frac{3}{2}(u_{xx}u_x^{-1}u_{xx}+u_{xx}u_x^{-1}u-uu_x^{-1}u_{xx}-uu_x^{-1}u)$$
$$u_{i,x}=\frac{1}{1-a_i^2}(u-a_iu_i)u_x^{-1}(a_iu-u_i)$$
$$(1-a_1^2)(u_1-a_2u_{12})(a_1u-u_1)^{-1}=(1-a_2^2)(u_2-a_1u_{12})(a_2u-u_2)^{-1}$$

These equations possess also 3-legs forms, generating nonabelian Toda lattices.

4.2 Yang-Baxter maps

The geometric construction of Yang-Baxter maps works also on the linear pencil of quadrics. Indeed, all points lie on the plane defined by the initial data X_1, X_2, X_3, so that $3D$-consistency is inherited from the planar situation. Nevertheless, the mapping itself cannot be reduced to the scalar one. Its general form is

$$X_{ij} = X_j + \frac{(a_i - a_j)(\langle X_j, SX_j\rangle + \langle s, X_j\rangle + \sigma)}{\langle X_i - X_j, (a_i S + T)(X_i - X_j)\rangle}(X_i - X_j)$$

where S, T are arbitrary symmetric matrices, s is an arbitrary vector and σ is an arbitrary scalar.

In [17], another examples of multifield Yang-Baxter maps were obtained by consideration of the interaction of matrix solitons with the non-trivial internal parameters (vector analog of phase shift).

[17] A.P. Veselov. Yang-Baxter maps and integrable dynamics. *Phys. Lett A* **314** (2003) 214–221.